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# Predictor Based on Complete Subgraphs for Consensus Problems with Communication Delay

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***Abstract***

*Consensus problems need communication between two or more agents. The existence of time delay in communication makes every agent not get the real-time states of the other agents. The main problem of delay system is the response starts slowing down and oscillating when the gain is increasing. This paper proposes predictor-feedback that reduces the effect of time delay. The predictor itself utilizes the complete subgraphs. Analytically the result generates faster response compared to the system without the predictor. Then, the proved solution is supported by a numerical solution.*

***Keywords:*** *consensus problems, delay, prediction, subgraphs*

**Abstrak**

Masalah konsensus membutuhkan komunikasi antara dua atau lebih agen. Adanya time delay dalam komunikasi membuat setiap agen tidak mendapatkan status real-time dari agen lain. Masalah utama sistem delay adalah respons mulai melambat dan berosilasi ketika penguatan meningkat. Makalah ini mengusulkan umpan balik prediktor yang mengurangi efek penundaan waktu. Prediktor itu sendiri menggunakan subgraf lengkap. Secara analitis hasilnya menghasilkan respons yang lebih cepat dibandingkan dengan sistem tanpa prediktor. Kemudian, solusi yang terbukti didukung oleh solusi numerik.

**Kata Kunci:** masalah konsensus, penundaan, prediksi, subgraf

## I. INTRODUCTION

Consensus problems become a fundamental problem in the cooperative control of multi-agent systems [1]-[3], such as flocking and formation control problems [4]. A group of agents, that are connected by a communication system, will solve their problems where the problems themself cannot be solved if these agents do not cooperate with each other [5]. The problem is widely studied as the LTI system furthermore also studied as non-linear systems such as autonomous robots [6]-[7].

The delayed LTI system also has been widely studied even though no exact analytic solution neither with Laplace transformation nor Lyapunov function. In other hand, special conditions such as symmetric matrices, normal matrices, and abelian groups are needed to be considered. The existance of delay is caused by physical's constraints such as air gap between the gears in mechanical gears system. The other example is signal transmissions between several satellites [8].

Research [1] explains that the consensus problem can be varied. One of them is presenting the lack of communication especially in the form of communication delay. The presence of delay in the consensus problem makes the roots of the characteristic equation are going to complex numbers and causing oscillation. As the delay goes larger, the oscillation goes bigger. In a certain boundary, the system becomes unstable. In addition, there is an upper bound of time delay that allows the system to become stable.

The delayed LTI system already has a solution that is called predictor-feedback [9]. It is open-loop and uses the knowledge of the LTI equation in integral form rather than differential equation to predict the future states of a system. The predictor itself needs all the states as

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information so it can't be used in consensus problems if there are 2 agents that aren't connected to each other.

In multi-agent system, the communications between the agents are represented in graph. The vertex is presenting agent and the edge is presenting communication between two agents. The predictor will work if the graph is a complete graph. Rather using the complete graph as the predictor, using the complete subgraph (part of the graph) can be tried to reduce the effect of delay. In the abelian matrices between non-delayed coefficient and delayed coefficient, it is already studied in [10] about how to improve the response of the system. Furthermore, [11] shows improvement in non-modified graphs by using modified feedback.

This paper contributes to how the delay-based predictor can be used in consensus problems even though not all the effects will be removed. Also, this paper will present the analytic result of two non-abelian symmetric matrices working in linear-exponent equation.

The organization of the paper is as follows. In Section II we set up the system and problem formulated. Section III presents predictor design. Section IV presents analytic results including comparison with the system without a predictor. Root-locus based numerical method, how if we add the gain, will be present in Section V And Section VI concludes the paper.

## II. Problem Formulation

Consider the model of each agent $i$ which is a single integrator model,

$$\dot{x}_i(t) = u(t), \quad i = 1,2,\dots,N \tag{1}$$

Those agents with the communication create a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ which the vertices $\mathcal{G}$ present the agents. We label each vertex as natural number from 1 to $N$ so the sets $\mathcal{V} = \{1,2,\dots,N\}$. Then pair of vertices $(i,j) \in \mathcal{E}$ if only if the agents $i$ and $j$ have a communication link. We said $(i,j)$ adjacent. Whenever agent $i$ send information to $j$ or vice versa there will be a time delay $\tau$. In this paper, $\tau$ was defines as communication delay.

The objective of consensus problem can be formulated for each $i$ and $j$ satisfy

$$\lim_{t\to\infty} x_i(t) - x_j(t) = 0 \tag{2}$$

Normally, if there is no predictor in (2) and use all of the connectivity. The system should be

$$\dot{x}(t) = -Lx(t-\tau) \tag{3}$$

which (3) explained widely in [3] where $x$ is state variable in the form of $N \times 1$ vector. The $N \times N$ matrix $L$ is the well-known Laplacian matrix.

## III. The Paper Formatting

In this paper, the main idea of designing the predictor is partitioning the graph $\mathcal{G}$ into some complete-subgraph, supposed $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_M$. We give an illustration on how to partition graphs in Fig 1.

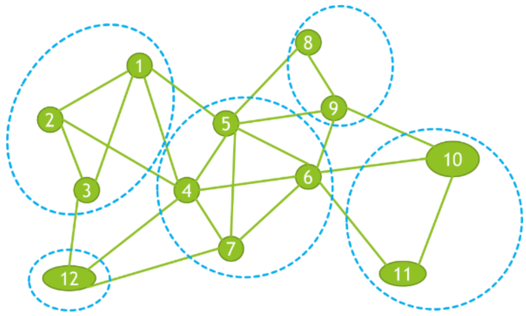


Fig. 1. Illustrative example of partitioning graph into several complete-subgraphs

Therefore, we define $L_\alpha$ as Laplacian matrix that contain the information of edges of $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_M$ while $L_\beta$ is the Laplacian matrix with edges excluded $L_\alpha$. Suppose

$$L_\alpha + L_\beta = L \tag{4}$$

and

$$L_\alpha = \begin{bmatrix} K_1 & 0 & \cdots & 0 \\ 0 & K_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_M \end{bmatrix} \tag{5}$$

where each $K_i$ is Laplacian matrix of complete-subgraph $\mathcal{G}_i$ for $i = 1,2,\dots,M$.

The reason why $L$ is needed to be partitioned into $L_\alpha$ dan $L_\beta$ is because in the predicting equation there will present a matrix $e^{-L}$. Yet, all of the term $l_{ij}$ from $L$ if $i$ is not connected to $j$ then $l_{ij} = 0$ but $[\mathbf{e}^{-\mathbf{L}}]_{ij} \neq 0$. The prediction couldn't be done in limited communication. Suppose the state variable as defined $\mathbf{x} = [x_1\ x_2 \dots x_N]^{\mathrm{T}}$. Here, the reasonable equation

$$\dot{x}(t) = -L_\alpha x(t) - L_\beta x(t-\tau) \tag{6}$$

Proceed from (4) we get the raw idea of predictor

$$x(t+\tau) = e^{-k_\alpha L_\alpha \tau} x(t) - \int_{-\tau}^{0} e^{k_\alpha L_\alpha \theta} L_\beta x(t+\theta) d\theta \tag{7}$$

Equation (6) need to be constructed to (1) and the constraint of communication delay. Consider of the time window, the agents measure the state and give the input at $t$ but receive the measured and predicted states from adjacent agents were delivered at $t-\tau$.

Equation (7) has variable states at $t+\tau$. It can be predicted using informations that have been delivered at $t-\tau$ until $t$. If the predicted states need the information from measured states, then they need the information from $t-\tau$ or earlier as we can see in Fig. 2. Thus, substitutes (7) into itself we get following equation.

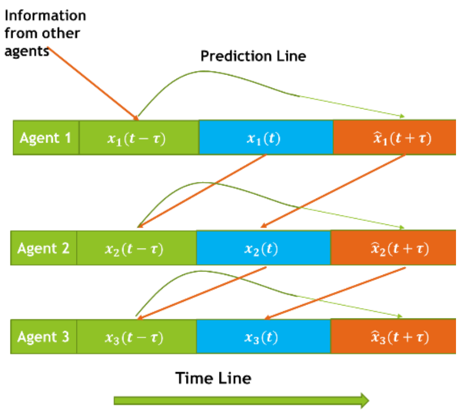


Fig. 2. Communication with Delay and State Prediction in each agent

$$\hat{x}(t+\tau) = e^{-L_\alpha\tau}\hat{x}(t) + \int_{-\tau}^{0} e^{L_\alpha\theta}\left(-L_\beta\hat{x}(t+\theta)\right)d\theta$$

$$= e^{-2L_\alpha\tau}x(t-\tau) - e^{-L_\alpha\tau}\int_{-\tau}^{0} e^{L_\alpha\theta}L_\beta x(t-\tau+\theta)d\theta - \int_{-\tau}^{0} e^{L\,\alpha\theta}L_\beta e^{-L_\alpha\tau}x(t-\tau+\theta)d\theta + \int_{-\tau}^{0}\int_{-\tau}^{0} e^{L_\alpha\theta}L_\beta e^{L_\alpha\phi}L_\beta x(t-\tau+\theta+\phi)\,d\phi\,d\theta \quad (8)$$

Then, considering each of the agent we have the prediction equation of each agent

$$\hat{x}_i(t+\tau) = \sum_{j=1}^{N}(r_{ij}(-2\tau)\hat{x}_j(t-\tau) - \int_{-\tau}^{0} s_{ij}(\theta-\tau,0)\,\hat{x}_j(t-\tau+\theta)d\theta - \int_{-\tau}^{0} s_{ij}(\theta,-\tau)\,\hat{x}_j(t-\tau)d\theta + \int_{-\tau}^{0}\int_{-\tau}^{0} s_{ij}(\theta,-\tau_c)\hat{u}_{\beta j}(t-\tau)\,d\phi\,d\theta$$

and the input for (1) become

$$u_i(t) = u_{\alpha i}(t) + u_{\beta i}(t) \quad (9)$$

$$u_{\alpha i}(t) = -L_{\alpha i}\hat{\hat{x}}(t) \quad (10)$$

$$u_{\beta i}(t) = -L_{\beta i}x(t-\tau) \quad (11)$$

where

$$r_{ij}(\theta) = \left(e^{k_\alpha L_\alpha\theta}\right)_{ij} \quad (12)$$

$$s_{ij}(\theta,\phi) = \left(e^{k_\alpha L_\alpha\theta}k_\beta L_\beta e^{k_\alpha L_\alpha\phi}\right)_{ij} \quad (13)$$

From the (9) to (13) we compose continuous and parallel prediction algorithm from each agent as follows

1. Measure $x_i(t)$
2. Receive $x_j(t-\tau)$ and $\hat{\hat{x}}_j(t)$ from adjacent agents
3. Send $u_{\alpha i}(t-\tau) = -\sum_{j=1}^{N} l_{\alpha ij}\hat{\hat{x}}_j(t)$
4. Send $u_{\beta i}(t-\tau) = -\sum_{j=1}^{N} l_{\beta ij}\hat{x}_j(t)$
5. Predict $\hat{u}_{\beta i}(t) = -k_\beta\sum_{j=1}^{N} l_{\beta ij}\hat{x}_j(t+\tau_{in})$
6. Predict $\hat{\hat{x}}_i(t+\tau)$

Send $\hat{\hat{x}}_i(t+\tau)$ , $x_i(t)$ dan $u_{\beta i}(t)$ to adjacent agents

## IV. Results and Discussion

In this section will discuss about the analytical results of the system with predictor compared to the system without the predictor. Then, the analiytical and numerical results of the system with predictor when the gain is varying. The next paragraph starts the first results and discussion.

**Define 1.** $\lambda_i^{\{M\}}$ are the eigenvalue of $M \in \mathbb{C}^{N\times N}$ dengan $\lambda_1^{\{M\}} \le \lambda_2^{\{M\}} \le \cdots \le \lambda_N^{\{M\}}$ associated with eigenvector $v_i^{\{M\}}$.

If $M$ diagonalizable then $V_M$, the eigenspace of $M$, is sets of eigenvectors $x$, then for $x \in V_M$ there exist $\lambda$ such that $Mx = \lambda x$.

We will prove the maximum $\tau^*$ such that the equation

$$\dot{x}(t) = -L_\alpha x(t) - L_\beta x(t-\tau) \quad (14)$$

for $\tau < \tau^*$ is stable is greater than the maximum $\tau^*$ from [1]

$$\dot{x}(t) = -L_\alpha x(t-\tau) - L_\beta x(t-\tau) \quad (15)$$

We know $\tau^*$ from (15)

$$\tau^* \le \frac{\pi}{2\lambda_N^{\{L_\alpha+L_\beta\}}}$$

The characteristic equation transform to (14)

$$\det\left(Is + L_\alpha + L_\beta e^{-s\tau}\right) = 0 \quad (16)$$

**Define 2.** $M_1(s,\tau) = Is + L_\alpha + L_\beta e^{-s\tau}$ and $s = \sigma + j\omega$ for $v \in \mathbb{C}^N - Span\{1_N\}$

Note for every normal matrix $M$ that we have

$$\lambda_1^{\{M+M^*\}} \le 2Re\left(\lambda_i^{\{M\}}\right) \le \lambda_N^{\{M+M^*\}}$$

with continuation, the smallest $\tau^*$ from (16) satisfy

$$M_1(j\omega,\tau^*)v = 0, \qquad v \notin Span\{1_N\}$$

$$\Rightarrow M_1(j\omega,\tau^*) + M_1^*(jw,\tau^*) = 2L_\alpha + 2L_\beta\cos\omega\tau^*$$

If $-\frac{\pi}{2} \le \omega\tau^* \le \frac{\pi}{2}$ then $\cos\omega\tau^* \ge 0$ and $2L_\alpha + L_\beta\cos\omega\tau^*$ positive semidefinite with $1_N$ is the only nullity. Contradiction. Then

$$|\omega\tau^*| > \frac{\pi}{2} \quad (17)$$

Consider

$$-jM_1(j\omega,\tau^*) + \left(-jM_1(jw,\tau^*)\right)^* = -jM_1(j\omega,\tau^*) + jM_1^*(jw,\tau^*) = 2I\omega - 2L_\beta\sin\omega\tau^*$$

analog with (17) we'll have

$$|\omega| \le \lambda_N^{\{L_\beta\}} \quad (18)$$

Since $L_\alpha + L_\beta - L_\beta = L_\alpha$ positive semi-definite then

$$|\omega| \le \lambda_N^{\{L_\beta\}} \le \lambda_N^{\{L_\alpha+L_\beta\}}$$

$$\Rightarrow \tau^* > \frac{\pi}{2|\omega|} \geq \frac{\pi}{2\lambda_N^{\{L_\beta\}}} \geq \frac{\pi}{2\lambda_N^{\{L_\alpha+L_\beta\}}} \quad (19)$$

Proved.

There is a special case where $L_\alpha$ and $L_\beta$ has some common eigenvector $v \notin Span\{1_N\}$ where

$$L_\alpha v = \lambda_\alpha v$$

$$L_\beta v = \lambda_\beta v$$

then $M_1(s,\tau)v = 0$ can be decomposed to

$$s + \lambda_\alpha + \lambda_\beta e^{-s\tau} = 0 \quad (20)$$

$$\Rightarrow s = \frac{W\left(\lambda_\beta e^{\lambda_\alpha \tau}\right)}{\tau} - \lambda_\alpha$$

with $W(\cdot)$ is Lambert-W Function. Same as previous, the smallest $\tau^*$ from $(20)$ satisfy $\sigma = 0$ then we separate the real and imaginary part from $(20)$

$$\lambda_\alpha = -\lambda_\beta \cos \omega\tau^* \quad (21)$$

$$\omega = \lambda_\beta \sin \omega\tau^* \quad (22)$$

$$\Rightarrow \omega^2 = \lambda_\beta^2 - \lambda_\alpha^2 \Rightarrow \omega = \pm\sqrt{\lambda_\beta^2 - \lambda_\alpha^2}$$

if $\lambda_\alpha \geq \lambda_\beta$ then it has no solution $\tau^*$, but if $\lambda_\beta > \lambda_\alpha$ then with the help of $(22)$ we get

$$\sin \omega\tau^* = \pm\frac{\sqrt{\lambda_\beta^2 - \lambda_\alpha^2}}{\lambda_\beta}$$

and from $(17)$ then

$$\omega\tau^* = \pm\pi \mp \arcsin\frac{\sqrt{\lambda_\beta^2 - \lambda_\alpha^2}}{\lambda_\beta} \Rightarrow \tau^* = \frac{\pi - \arcsin\frac{\sqrt{\lambda_\beta^2 - \lambda_\alpha^2}}{\lambda_\beta}}{\sqrt{\lambda_\beta^2 - \lambda_\alpha^2}}$$

$$\tau^* = \frac{\pi - \arcsin\frac{\sqrt{\lambda_\beta^2-\lambda_\alpha^2}}{\lambda_\beta}}{\sqrt{\lambda_\beta^2-\lambda_\alpha^2}} > \frac{\pi}{2\lambda^{\{L_\alpha+L_\beta\}}} \quad (23)$$

The next discussion will talk about the root's behaviour when the gains are varying in the both $L_\beta$ and $L_\alpha$ and analyze them separately. First, here the result if $L_\beta$ is added by gain $k_\beta$ and the equation becomes

$$\dot{x}(t) = -L_\alpha x(t) - k_\beta L_\beta x(t-\tau)$$

or in Laplace transformation

$$\det\left(s + L_\alpha + k_\beta L_\beta e^{-s\tau}\right) = 0 \quad (24)$$

Recall $(19)$ and then modified it to

$$k_\beta < \frac{\pi}{2\lambda_N^{\{L_\beta\}}\tau}$$

**Assumption.** Equation $(24)$ is a stable consensus.

**Lemma 1**. We will prove there will be $P > 0$, hence for $k_\beta < P$ , equation $(24)$ will have real solution $s$.

Prove : Suppose $s = \sigma + j\omega$ with $\omega \neq 0$ then consider

$$-jM_1(s,\tau) + \left(-jM_1(s,\tau)\right)^* = 2I\omega - 2k_\beta L_\beta \sin\omega\tau\, e^{-\sigma\tau}$$

and the largest eigenvalue of $L_\beta$ then

$$|\omega| < |\sin\omega\tau| k_\beta \lambda_N^{\{L_\beta\}} e^{-\sigma\tau} < |\omega\tau| k_\beta \lambda_N^{\{L_\beta\}} e^{-\sigma\tau}$$

$$\Rightarrow 1 < \tau k_\beta \lambda_N^{\{L_\beta\}} e^{-\sigma\tau}$$

$$\sigma \leq \frac{\ln\left(\tau k_\beta \lambda_N^{\{L_\beta\}}\right)}{\tau} \quad (25)$$

When $k_\beta = 0$ we know $s$ will be the eigenvalue of $-L_\alpha$ which are non positive real numbers. By continuity, if $k_\beta \to 0$ then $\sigma \to \sigma_0$ for $\sigma_0$ is eigenvalue from $-L_\alpha$ but

$$\sigma \leq \frac{\ln\left(\tau k_\beta \lambda_N^{\{L_\beta\}}\right)}{\tau} \to -\infty$$

is a contradiction and the Lemma is proved.

On the other hand, when $s$ is real number, using [15] and because $L_\alpha$ and $L_\beta$ both are positive semidefinite then

$$L_\alpha e^{-s\tau} + k_\beta L_\beta e^{-s\tau} > L_\alpha + k_\beta L_\beta e^{-s\tau} > k_\beta L_\beta e^{-s\tau}$$

$$\Rightarrow \lambda_i^{\{L_\alpha\}} e^{-s\tau} + k_\beta \lambda_N^{\{L_\beta\}} e^{-s\tau} > \lambda_i^{\{L_\alpha + k_\beta L_\beta\}} e^{-s\tau} > -s > \lambda_i^{\{L_\alpha\}} + k_\beta \lambda_1^{\{L_\beta\}} e^{-s\tau} \quad \dots (26)$$

Now, we move to the case when $s$ is not a real number.

**Lemma 2**. If $|\omega| > 0$ then $k_\beta > \frac{e^{-1}}{\tau\lambda_N^{\{L_\beta\}}}$.

Prove: Consider the following equation

$$jM_1(s,\tau) + \left(-jM_1(s,\tau)\right)^* = 2I\sigma - 2k_\beta L_\beta \cos\omega\tau\, e^{-\sigma\tau} - 2L_\alpha$$

$$-\sigma < k_\beta \lambda_N^{\{L_\beta\}} \cos\omega^+\tau\, e^{-\sigma\tau} \leq k_\beta \lambda_N^{\{L_\beta\}} e^{-\sigma\tau}$$

$$-\tau k_\beta \lambda_N^{\{L_\beta\}} < \sigma\tau e^{\sigma\tau}$$

if $\tau k_\beta \lambda_N^{\{L_\beta\}} \leq e^{-1}$ thus $-\sigma\tau < e^{-1}$ and using The Lambert W-function

$$\frac{W_0\left(-\tau k_\beta \lambda_N^{\{L_\beta\}}\right)}{\tau} < \sigma$$

$$\Rightarrow W_0\left(-\tau k_\beta \lambda_N^{\{L_\beta\}}\right) < \ln\left(\tau k_\beta \lambda_N^{\{L_\beta\}}\right)$$

$$\Rightarrow e^{W_0\left(-\tau k_\beta \lambda_N^{\{L_\beta\}}\right)} < e^{\ln\left(\tau k_\beta \lambda_N^{\{L_\beta\}}\right)}$$

$$\Rightarrow \frac{-\tau k_\beta \lambda_N^{\{L_\beta\}}}{W_0\left(-\tau k_\beta \lambda_N^{\{L_\beta\}}\right)} < \tau k_\beta \lambda_N^{\{L_\beta\}}$$

$$\Rightarrow W_0\left(-\tau k_\beta \lambda_N^{\{L_\beta\}}\right) < -1$$

which is a contradiction then

$$\tau k_\beta \lambda_N^{\{L_\beta\}} > e^{-1}$$

$$\Rightarrow k_\beta > \frac{e^{-1}}{\tau \lambda_N^{\{L_\beta\}}} \quad (26)$$

From $(24) - (26)$ brief some summary

1. For $k_\beta \leq \frac{\pi}{2\lambda_N^{\{L_\beta\}}\tau}$ , the system will guarantee consensus stability.
2. When $k_\beta$ is small enough it is guaranteed to have $N-1$ negative real roots.
3. When $k_\beta$ is small enough it is guaranteed to have $N-M$ real root $s$ which is less than $\lambda_i^{\{L_\alpha\}}$ which is 0.
4. When $k_\beta \leq \frac{e^{-1}}{\tau \lambda_N^{\{L_\beta\}}}$, the system is guaranteed to have $N-1$ negative real roots.

In the next discussion is going to about giving the gain in the predicted component.

$$\dot{x}(t) = -k_\alpha L_\alpha x(t) - L_\beta x(t-\tau)$$

or the charateristic equation as follows:

$$M = Is + k_\alpha L_\alpha + L_\beta e^{-s\tau} \quad (27)$$

Because $L_\alpha$ normal then there is a unitary $U$ thus

$$U^* L_\alpha U = D_\alpha = \begin{bmatrix} \lambda_1^{\{L\alpha\}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{\{L_\alpha\}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N^{\{L_\alpha\}} \end{bmatrix}$$

with $D_\alpha$ diagonal matrix containing $N-M$ eigenvalue positive and $M$ eigenvalue 0, then $\det(M) = 0 \Leftrightarrow \det(U^* M U) = 0$. Substitute to $(27)$

$$U^* M U = Is + k_\alpha D_\alpha + U^* L_\beta U e^{-s\tau}$$

Suppose that $\left[U^* L_\beta U\right]_{ij} = a_{ij}$ and

$$r_i = \sum_{j=1, j\neq i}^{N} |a_{ij}|$$

with Gersgorin's theorem,

$$\bigcup_{i=1}^{n} \left\{ s \in \mathbb{C} : \left| s + k_\alpha \lambda_i^{\{L\alpha\}} + a_{ii} e^{-s\tau} \right| \leq r_i e^{-\sigma\tau} \right\}$$

and look at the real part,

$$\bigcup_{i=1}^{n} \left\{ s \in \mathbb{C} : \left| \sigma + k_\alpha \lambda_i^{\{L\alpha\}} + a_{ii} e^{-\sigma\tau} \cos \omega\tau \right| \leq r_i e^{-\sigma\tau} \right\}$$

If $i > M$ , thus $\lambda_i^{\{L\alpha\}} > 0$ and then

$$\sigma + k_\alpha \lambda_i^{\{L\alpha\}} + a_{ii} e^{-\sigma\tau} \cos \omega\tau \leq r_i e^{-\sigma\tau}$$

$$\Rightarrow \sigma + k_\alpha \lambda_i^{\{L\alpha\}} \leq (r_{ii} + |a_{ii}|) e^{-\sigma\tau}$$

$$\Rightarrow \sigma - (r_{ii} + |a_{ii}|) e^{-\sigma\tau} \leq -k_\alpha \lambda_i^{\{L\alpha\}}$$

Because $\sigma - (r_{ii} + |a_{ii}|) e^{-\sigma\tau}$ is function in the term of $\sigma$, then for each $P < 0$ there is such a $k_\alpha$ that

$$\sigma < P \quad (28)$$

but for $i \leq M, \lambda_i^{\{L\alpha\}} = 0$ , then

$$\bigcup_{i=1}^{M} \left\{ s \in \mathbb{C} : \left| s + k_\alpha \lambda_i^{\{L\alpha\}} + a_{ii} e^{-s\tau} \right| \leq r_i e^{-\sigma\tau} \right\}$$

$$= \bigcup_{i=1}^{n} \{ s \in \mathbb{C} : |s + + a_{ii} e^{-s\tau}| \leq r_i e^{-\sigma\tau} \}$$

It does not change for any value $k_\alpha$ then for a large enough $k_\alpha$,

$$\bigcup_{i=1}^{M} \{ s \in \mathbb{C} : |s + a_{ii} e^{-s\tau}| \leq r_i e^{-\sigma\tau} \} \quad (29)$$

and

$$\bigcup_{i=M+1}^{N} \left\{ s \in \mathbb{C} : \left| s + k_\alpha \lambda_i^{\{L\alpha\}} + a_{ii} e^{-s\tau} \right| \leq r_i e^{-\sigma\tau} \right\} \quad (30)$$

The sets $(29)$ and $(30)$ will be disjointed. We brief some summary from $(27) - (30)$.

1. For each $P < 0$ there is $k_\alpha > 0$ hence that $N - M$ roots have a real part that is less than $P$.
2. There are $M$ sets that are invariant to $k_\alpha$.
3. While the other $N - M$ root in the sets $(30)$ will diverge to $-\infty$ when $k_\alpha \to +\infty$

From $(24) - (30)$ , it was also concluded that gain $k_\alpha > 0$ will be invariant with the stability of the system.

Root locus in this study was used to verify the results of analytical calculation. The basis of the algorithm is in [16] which is based on newton algorithms. In addition, it is necessary to define the equations and variables. There are two equations that will be verified with this algorithm:

$$M(k,s) = Is + L_\alpha + kL_\beta e^{-s\tau} \quad (31)$$

$$M(k,s) = Is + kL_\alpha + L_\beta e^{-s\tau} \quad (32)$$

Here the results $M(k,s)v(s) = 0$ from $M(k,s)$ in $(31)$ with $k$ valued from 0 to 10 with interval 0.01 on graphs with

$$L_\alpha = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$L_\beta = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

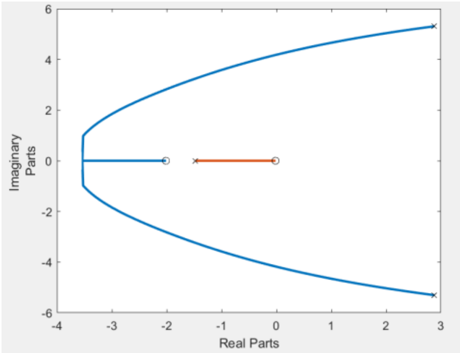


Fig. 3. Root Locus with $M(k,s) = Is + L_\alpha + kL_\beta e^{-s\tau}$

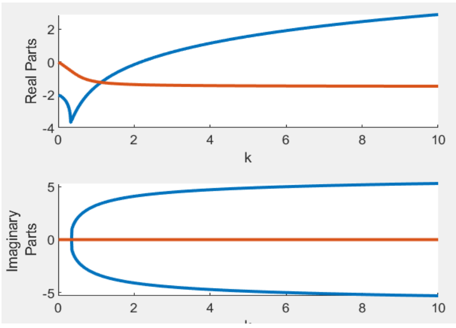


Fig. 4. Real and Imaginary Parts of $\boldsymbol{s}$ in the term of $\boldsymbol{k}$

It is seen in Fig. 3 and 4 that there is a value of $s = \sigma + j\omega$ which is blue, at first $\omega = 0$ and $\sigma$ shrinks in the real domain then $|\omega|$ increases and $\sigma$ grows to make it unstable.

Here are the root locus results of some roots from (32) with $k$ valued from 0 to 10 with intervals of 0.01 on graphs with

$$L_\alpha = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

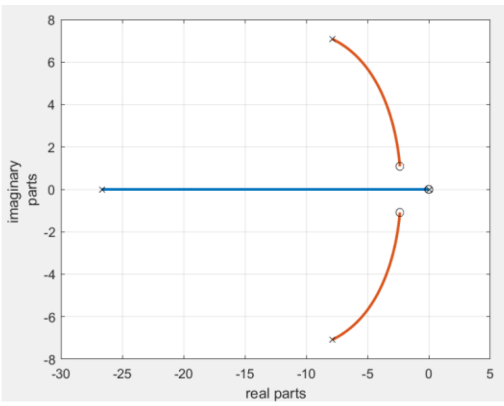


Fig. 5. Root Locus with $M(k,s) = Is + kL_\alpha + L_\beta e^{-s\tau}$

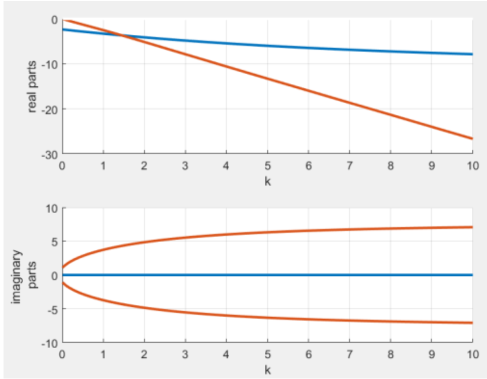


Fig. 6. Real and Imaginary Parts of $s$ in the term of $k$

There should be 4-1=3 roots visible in the locus. However, because there is an eigenvector which is an eigenvector $L_\alpha$ and $L_\beta$ are $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ that contains twin roots resulting in only 2 roots visible. It is seen that in Fig. 5 and Fig. 6 the blue color present (29) while the orange presents (30).

## V. Conclusion

This manuscript solved consensus problems with fixed and unchanging input delays and communication delays over time by the identical agents in the form of a linear system in general by numerical solution. This paper also contributes to how the delay-based predictor can be used in consensus problems even though not all the effects will be removed. Root locus in this study is used to verify the results of analytical calculations. Analytically, the result generates faster response compared to the system without the predictor. And if all the information from each agent is sent to each agent, then whatever the delay (time independent) can stabilize consensus with the system response proportion to the system gain. The analytic result of two non-abelian symmetric matrices working in linear-exponent equation. Finally, numerical solution has been presented to demonstrate the logic of the proposed theoretical findings

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